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Information-Geometric Indicators of Chaos in Gaussian Models on Statistical Manifolds of Negative Ricci Curvature

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Abstract A new information-geometric approach to chaotic dynamics on curved statistical manifolds based on Entropic Dynamics (ED) is proposed. It is shown that the hyperbolicity of a non-maximally symmetric $6N$-dimensional statistical manifold $\mathcal{M}_s$ underlying an ED Gaussian model describing an arbitrary system of $3N$ degrees of freedom leads to linear information-geometric entropy growth and to exponential divergence of the Jacobi vector field intensity, quantum and classical features of chaos respectively.

Keywords Inductive inference · Information geometry · Statistical manifolds · Entropy · Nonlinear dynamics · Chaos

1 Introduction

The unification of classical theory of gravity with quantum theories of electromagnetic, weak and strong forces is one of the major problems in modern physics. Entropic Dynamics (ED) [1], namely the combination of principles of inductive inference (Maximum relative Entropy Methods, ME methods) [2–4] and methods of Information Geometry (Riemannian geometry applied to probability theory, IG) [5], is a theoretical framework constructed on statistical manifolds and it is developed to investigate the possibility that laws of physics, either classical or quantum, might reflect laws of inference rather than laws of nature. Examples of dynamics that can be deduced from principles of probable inference are not absent in physics. The theory of thermodynamics [6, 7] and to a certain degree, quantum mechanics [8, 9], are examples of fundamental physical theories that could be derived from general principles of inference. In constructing an ED-model, the first step is to identify the appropriate variables (relevant information) describing the system and thus the corresponding space of macrostates. This is the most delicate step because there is no systematic way to

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search for the right macro variables; it is a matter of intuition, trial and error. Once the information and normalization constraints are identified, using ME methods, the probability distribution characterizing the system can be computed. Finally, using IG methods, a Fisher-Rao information metric [10, 11] can be assigned to the space of macrostates of the system. Given the Fisher-Rao information metric, the geometric structure of the manifold underlying the ED can be studied in detail: metric tensor, Christoffel connection coefficients, Ricci and Riemann statistical curvature tensors, sectional and Ricci scalar curvatures, Jacobi and Killing fields can be calculated. ME methods are inductive inference tools. They are used for updating from a prior to a posterior distribution when new information in the form of constraints becomes available. Basically, information is processed using ME methods in the framework of Information Geometry. The ED model follows from an assumption about what information is relevant to predict the evolution of the system. In this work, we focus only on reversible aspects of the ED model. In this case, given a known initial macrostate and that the system evolves to a final known macrostate, we investigate the possible trajectories of the system. Given two probability distributions, a notion of "distance" between them is provided by IG.

In this paper, our objective is to report some relevant results obtained in the realm of chaos theory (hypersensitivity to initial conditions) using the ED formalism. It is known there is not a well defined unifying characterization of chaos in classical and quantum physics [12, 13]. In the Riemannian Geometric Approach [14, 15] to classical chaos, the search for a link between the Jacobi field intensity and the Ricci (sectional) curvature of the dynamical manifold is under investigation [16]. In the Zurek-Paz criterion of quantum chaos [17, 18], instead, the search for a potential link between the linearity of the entropy growth and the curvature of the dynamical manifold underlying chaotic systems is still open. In our information-geometric approach, it is shown that these three indicators of chaos (Curvature-Jacobi Field Intensity-Entropy) are linked: the hyperbolicity of a $6N$-dimensional statistical manifold $\mathcal{M}_s$ underlying an ED Gaussian model leads to linear information-geometric entropy growth and to exponential divergence of the Jacobi vector field intensity, quantum and classical features of chaos respectively. We assume that the arbitrary (physical-biological) system under investigation has $3N$ degrees of freedom $\{x_a^{(\alpha)}\}_{a=1,2,3}^{\alpha=1,\ldots,N}$, each one described by two pieces of relevant information, its expectation value $\langle x_a^{(\alpha)}\rangle$ and variance $\Delta x_a^{(\alpha)} \stackrel{\text{def}}{=} \sqrt{\langle (x_a^{(\alpha)} - \langle x_a^{(\alpha)}\rangle)^2\rangle}$. This leads to consider an ED model on a $6N$-dimensional statistical manifold $\mathcal{M}_s$. First, we show that $\mathcal{M}_s$ has a constant negative Ricci curvature proportional to the number of degrees of freedom of the system, $\mathcal{R}_{\mathcal{M}_s} = -3N$. Second, we suggest the information-geometric analog of the Zurek-Paz quantum chaos criterion. It is shown that the system explores statistical volume elements on $\mathcal{M}_s$ at an exponential rate. We define an information-geometric entropy (IGE) of the system, $\mathcal{S}_{\mathcal{M}_s}$. We show that $\mathcal{S}_{\mathcal{M}_s}$ increases linearly in time (statistical evolution parameter) and it is proportional to the number of degrees of freedom of the system and to the information-geometric analogue of the Lyapunov exponents [19]. Finally, we show that the geodesics on the manifold $\mathcal{M}_s$ are described by hyperbolic trajectories. Using the Jacobi-Levi-Civita (JLC) equation for geodesic spread, it is shown that the intensity of the Jacobi vector field intensity $J_{\mathcal{M}_S}$ diverges exponentially (standard feature of classical chaos) and it is proportional to the number of degrees of freedom of the system. In conclusion, the Ricci scalar curvature $\mathcal{R}_{\mathcal{M}_s}$, the information-geometric entropy $\mathcal{S}_{\mathcal{M}_s}$ and the Jacobi vector field intensity $J_{\mathcal{M}_S}$ are proportional to the number of Gaussian-distributed microstates of the system. The relevance of this proportionality will be discussed in some detail in Sect. 3 of this article.

The layout of this paper is as follows. In Sect. 2, we describe the ED Gaussian model being studied. In Sect. 3, we introduce the information-geometric indicators of chaos for our theoretical model. Finally, in Sect. 4 we present our final remarks.

2 Entropic Dynamical Gaussian Model

We consider an ED model whose microstates span a $3N$-dimensional space labelled by the variables $\{\vec{X}\} = \{\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(N)}\}$ with $\vec{x}^{(\alpha)} \equiv (x_1^{(\alpha)}, x_2^{(\alpha)}, x_3^{(\alpha)})$, $\alpha = 1, \ldots, N$ and $x_a^{(\alpha)} \in \mathbb{R}$ with $a = 1, 2, 3$. We assume the only testable information pertaining to the quantities $x_a^{(\alpha)}$ consists of the expectation values $\langle x_a^{(\alpha)} \rangle$ and the variance $\Delta x_a^{(\alpha)}$. The set of these expected values define the $6N$-dimensional space of macrostates of the system. A measure of distinguishability among the *macrostates* of the ED model is achieved by assigning a probability distribution $P(\vec{X}|\vec{\Theta})$ to each $6N$-dimensional macrostate $\vec{\Theta} \overset{\text{def}}{=} \{({}^{(1)}\theta_a^{(\alpha)}, {}^{(2)}\theta_a^{(\alpha)})\}_{3N\text{-pairs}} = \{(\langle x_a^{(\alpha)} \rangle, \Delta x_a^{(\alpha)})\}_{3N\text{-pairs}}$ with $\alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. The process of assigning a probability distribution to each state provides $\mathcal{M}_S$ with a metric structure. Specifically, the Fisher-Rao information metric defined in (7) is a measure of distinguishability among macrostates. It assigns an IG to the space of states. Consider an arbitrary physical system evolving over a $3N$-dimensional space. The variables $\{\vec{X}\} \overset{\text{def}}{=} \{\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(N)}\}$ label the $3N$-dimensional space of microstates of the system. We assume that all information relevant to the dynamical evolution of the system is contained in the probability distributions. For this reason, no other information is required. Each macrostate may be thought as a point of a $6N$-dimensional statistical manifold with coordinates given by the numerical values of the expectations ${}^{(1)}\theta_a^{(\alpha)}$ and ${}^{(2)}\theta_a^{(\alpha)}$. The available information can be written in the form of the following $6N$ information constraint equations,

$$\begin{aligned} \langle x_a^{(\alpha)} \rangle &= \int_{-\infty}^{+\infty} dx_a^{(\alpha)} x_a^{(\alpha)} P_a^{(\alpha)}\left(x_a^{(\alpha)} \middle| {}^{(1)}\theta_a^{(\alpha)}, {}^{(2)}\theta_a^{(\alpha)}\right), \\ \Delta x_a^{(\alpha)} &= \left[\int_{-\infty}^{+\infty} dx_a^{(\alpha)} \left(x_a^{(\alpha)} - \langle x_a^{(\alpha)} \rangle\right)^2 P_a^{(\alpha)}\left(x_a^{(\alpha)} \middle| {}^{(1)}\theta_a^{(\alpha)}, {}^{(2)}\theta_a^{(\alpha)}\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{1}$$

The probability distributions $P_a^{(\alpha)}$ in (1) are constrained by the conditions of normalization,

$$\int_{-\infty}^{+\infty} dx_a^{(\alpha)} P_a^{(\alpha)}\left(x_a^{(\alpha)} \middle| {}^{(1)}\theta_a^{(\alpha)}, {}^{(2)}\theta_a^{(\alpha)}\right) = 1. \tag{2}$$

Information theory identifies the Gaussian distribution as the maximum entropy distribution if only the expectation value and the variance are known [20]. ME methods [2–4] allow us to associate a probability distribution $P(\vec{X}|\vec{\Theta})$ to each point in the space of states $\vec{\Theta}$. The distribution that best reflects the information contained in the prior distribution $m(\vec{X})$ updated by the information $(\langle x_a^{(\alpha)} \rangle, \Delta x_a^{(\alpha)})$ is obtained by maximizing the relative entropy

$$S(\vec{\Theta}) = -\int d^{3N}\vec{X} P(\vec{X}|\vec{\Theta}) \log\left(\frac{P(\vec{X}|\vec{\Theta})}{m(\vec{X})}\right), \tag{3}$$

where $m(\vec{X})$ is the prior probability distribution. As a working hypothesis, the prior $m(\vec{X})$ is set to be uniform since we assume the lack of prior available information about the system (postulate of equal a priori probabilities). Upon maximizing (3), given the constraints (1) and (2), we obtain

$$P(\vec{X}|\vec{\Theta}) = \prod_{\alpha=1}^{N}\prod_{a=1}^{3} P_a^{(\alpha)}\left(x_a^{(\alpha)}\middle|\mu_a^{(\alpha)}, \sigma_a^{(\alpha)}\right) \tag{4}$$

where

$$P_a^{(\alpha)}\left(x_a^{(\alpha)}\middle|\mu_a^{(\alpha)}, \sigma_a^{(\alpha)}\right) = \left(2\pi\left[\sigma_a^{(\alpha)}\right]^2\right)^{-\frac{1}{2}} \exp\left[-\frac{(x_a^{(\alpha)} - \mu_a^{(\alpha)})^2}{2(\sigma_a^{(\alpha)})^2}\right] \tag{5}$$

and, *in standard notation for Gaussians*, ${}^{(1)}\theta_a^{(\alpha)} \stackrel{\text{def}}{=} \langle x_a^{(\alpha)}\rangle \equiv \mu_a^{(\alpha)}$, ${}^{(2)}\theta_a^{(\alpha)} \stackrel{\text{def}}{=} \Delta x_a^{(\alpha)} \equiv \sigma_a^{(\alpha)}$. The probability distribution (4) encodes the available information concerning the system. Note that we have assumed uncoupled constraints among microvariables $x_a^{(\alpha)}$. In other words, we assumed that information about correlations between the microvariables need not to be tracked. This assumption leads to the simplified product rule (4). However, coupled constraints would lead to a generalized product rule in (4) and to a metric tensor (7) with non-trivial off-diagonal elements (covariance terms). Correlation terms may be fictitious. They may arise for instance from coordinate transformations. On the other hand, correlations may arise from external fields in which the system is immersed. In such situations, correlations among $x_a^{(\alpha)}$ effectively describe interaction between the microvariables and the external fields. Such generalizations would require more delicate analysis.

We cannot determine the evolution of microstates of the system since the available information is insufficient. Not only is the information available insufficient but we also do not know the equation of motion. In fact there is no standard "equation of motion". Instead we can ask: how close are the two total distributions with parameters $(\mu_a^{(\alpha)}, \sigma_a^{(\alpha)})$ and $(\mu_a^{(\alpha)} + d\mu_a^{(\alpha)}, \sigma_a^{(\alpha)} + d\sigma_a^{(\alpha)})$? Once the states of the system have been defined, the next step concerns the problem of quantifying the notion of change from the *macrostate* $\vec{\Theta}$ to the *macrostate* $\vec{\Theta} + d\vec{\Theta}$. A convenient measure of change is distance. The measure we seek is given by the dimensionless "distance" ds between $P(\vec{X}|\vec{\Theta})$ and $P(\vec{X}|\vec{\Theta} + d\vec{\Theta})$,

$$ds^2 = g_{\mu\nu} d\Theta^\mu d\Theta^\nu \tag{6}$$

where

$$g_{\mu\nu} = \int d\vec{X}\, P(\vec{X}|\vec{\Theta}) \frac{\partial \log P(\vec{X}|\vec{\Theta})}{\partial\Theta^\mu} \frac{\partial \log P(\vec{X}|\vec{\Theta})}{\partial\Theta^\nu} \tag{7}$$

is the Fisher-Rao metric [10, 11]. Substituting (4) into (7), the metric $g_{\mu\nu}$ on $\mathcal{M}_s$ becomes a $6N \times 6N$ matrix M made up of $3N$ blocks $M_{2\times 2}$ with dimension 2×2 given by,

$$M_{2\times 2} = \begin{pmatrix} (\sigma_a^{(\alpha)})^{-2} & 0 \\ 0 & 2 \times (\sigma_a^{(\alpha)})^{-2} \end{pmatrix} \tag{8}$$

with $\alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. From (7), the "length" element (6) reads,

$$ds^2 = \sum_{\alpha=1}^{N}\sum_{a=1}^{3}\left[\frac{1}{(\sigma_a^{(\alpha)})^2} d\mu_a^{(\alpha)2} + \frac{2}{(\sigma_a^{(\alpha)})^2} d\sigma_a^{(\alpha)2}\right]. \tag{9}$$

We bring attention to the fact that the metric structure of $\mathcal{M}_s$ is an emergent (not fundamental) structure. It arises only after assigning a probability distribution $P(\vec{X}|\vec{\Theta})$ to each state $\vec{\Theta}$.

3 Information-Geometric Indicators of Chaos

In this section, we introduce the relevant indicators of chaoticity within our theoretical formalism. They are the Ricci scalar curvature $\mathcal{R}_{\mathcal{M}_s}$ (or, more correctly, the sectional curvature $\mathcal{K}_{\mathcal{M}_S}$ [21]), the Jacobi vector field intensity $J_{\mathcal{M}_S}$ and the IGE $\mathcal{S}_{\mathcal{M}_s}$.

3.1 Ricci Scalar Curvature

Given the Fisher-Rao information metric, we use standard differential geometry methods applied to the space of probability distributions to characterize the geometric properties of $\mathcal{M}_s$. Recall that the Ricci scalar curvature $\mathcal{R}$ is given by,

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}, \tag{10}$$

where $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$ so that $g^{\mu\nu} = (g_{\mu\nu})^{-1}$. The Ricci tensor $R_{\mu\nu}$ is given by,

$$R_{\mu\nu} = \partial_{\varepsilon}\Gamma^{\varepsilon}_{\mu\nu} - \partial_{\nu}\Gamma^{\varepsilon}_{\mu\varepsilon} + \Gamma^{\varepsilon}_{\mu\nu}\Gamma^{\eta}_{\varepsilon\eta} - \Gamma^{\eta}_{\mu\varepsilon}\Gamma^{\varepsilon}_{\nu\eta}. \tag{11}$$

The Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ appearing in the Ricci tensor are defined in the standard way,

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\varepsilon}(\partial_{\mu} g_{\varepsilon\nu} + \partial_{\nu} g_{\mu\varepsilon} - \partial_{\varepsilon} g_{\mu\nu}). \tag{12}$$

Using (9) and the definitions given above, we can show that the Ricci scalar curvature becomes

$$\mathcal{R}_{\mathcal{M}_s} = -3N < 0. \tag{13}$$

From (13) we conclude that $\mathcal{M}_s$ is a $6N$-dimensional statistical manifold of constant negative Ricci scalar curvature. A detailed analysis on the calculation of Christoffel connection coefficients using the ED formalism can be found in [22–24]. Furthermore, it can be shown that $\mathcal{M}_s$ is not a pseudosphere (maximally symmetric manifold) since its sectional curvature is not constant. As a final remark, we emphasize that the negativity of the Ricci scalar $\mathcal{R}_{\mathcal{M}_S}$ implies the existence of expanding directions in the configuration space manifold $\mathcal{M}_s$. Indeed, since $\mathcal{R}_{\mathcal{M}_S}$ is the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements [21], the negativity of the Ricci scalar is only a sufficient (not necessary) condition for local instability of geodesic flow. For this reason, the negativity of the scalar provides a strong criterion of local instability.

3.2 Information-Geometrodynamical Entropy

At this point, we study the trajectories of the system on $\mathcal{M}_s$. We emphasize ED can be derived from a standard principle of least action (Maupertuis-Euler-Lagrange-Jacobi-type) [1, 25]. The main differences are that the dynamics being considered here, namely Entropic Dynamics, is defined on a space of probability distributions $\mathcal{M}_s$, not on an ordinary linear space V and the standard coordinates q_{μ} of the system are replaced by statistical macrovariables Θ^{μ}. The geodesic equations for the macrovariables of the Gaussian ED model are given by [26],

$$\frac{d^2\Theta^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\rho}\frac{d\Theta^{\nu}}{d\tau}\frac{d\Theta^{\rho}}{d\tau} = 0 \tag{14}$$

with $\mu = 1, 2, \ldots, 6N$. Observe that the geodesic equations are *nonlinear*, second order coupled ordinary differential equations. These equations describe a dynamics that is reversible and their solution is the trajectory between an initial and a final macrostate. The trajectory can be equally well traversed in both directions. We seek the explicit form of (14) for the pairs of statistical coordinates $(\mu_a^{(\alpha)}, \sigma_a^{(\alpha)})$. Substituting the explicit expression of the Christoffel connection coefficients into (14), the geodesic equations for the macrovariables $\mu_a^{(\alpha)}$ and $\sigma_a^{(\alpha)}$ associated to the microstate $x_a^{(\alpha)}$ become,

$$\frac{d^2\mu_a^{(\alpha)}}{d\tau^2} - \frac{2}{\sigma_a^{(\alpha)}}\frac{d\mu_a^{(\alpha)}}{d\tau}\frac{d\sigma_a^{(\alpha)}}{d\tau} = 0, \qquad \frac{d^2\sigma_a^{(\alpha)}}{d\tau^2} - \frac{1}{\sigma_a^{(\alpha)}}\left(\frac{d\sigma_a^{(\alpha)}}{d\tau}\right)^2 + \frac{1}{2\sigma_a^{(\alpha)}}\left(\frac{d\mu_a^{(\alpha)}}{d\tau}\right)^2 = 0. \tag{15}$$

with $\alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. This is a set of coupled ordinary differential equations, whose solutions are

$$\mu_a^{(\alpha)}(\tau) = \frac{\frac{(B_a^{(\alpha)})^2}{2\beta_a^{(\alpha)}}}{\exp(-2\beta_a^{(\alpha)}\tau) + \frac{(B_a^{(\alpha)})^2}{8(\beta_a^{(\alpha)})^2}}, \qquad \sigma_a^{(\alpha)}(\tau) = \frac{B_a^{(\alpha)}\exp(-\beta_a^{(\alpha)}\tau)}{\exp(-2\beta_a^{(\alpha)}\tau) + \frac{(B_a^{(\alpha)})^2}{8(\beta_a^{(\alpha)})^2}} + C_a^{(\alpha)}. \tag{16}$$

The quantities $B_a^{(\alpha)}$, $C_a^{(\alpha)}$, $\beta_a^{(\alpha)}$ are real integration constants and they can be evaluated once the boundary conditions are specified. We observe that since every geodesic is well-defined for all temporal parameters τ, $\mathcal{M}_s$ constitutes a geodesically complete manifold [27]. It is therefore a natural setting within which one may consider global questions and search for a weak criterion of chaos [15]. Furthermore, since $|\mu_a^{(\alpha)}(\tau)| < +\infty$ and $|\sigma_a^{(\alpha)}(\tau)| < +\infty \forall \tau \in \mathbb{R}^+$, $\forall a = 1, 2, 3$ and $\forall \alpha = 1, \ldots, N$, the parameter space $\{\vec{\Theta}\}$ (homeomorphic to $\mathcal{M}_s$) is compact. The compactness of the configuration space manifold $\mathcal{M}_s$ assures the folding mechanism of information-dynamical trajectories (the folding mechanism is a key-feature of true chaos, [15]).

We are interested in investigating the stability of the trajectories of the ED model considered on $\mathcal{M}_s$. It is known [25] that the Riemannian curvature of a manifold is closely connected with the behavior of the geodesics on it. If the Riemannian curvature of a manifold is negative, geodesics (initially parallel) rapidly diverge from one another. For the sake of simplicity, we assume very special initial conditions: $B_a^{(\alpha)} \equiv \Lambda$, $\beta_a^{(\alpha)} \equiv \lambda \in \mathbb{R}^+$, $C_a^{(\alpha)} = 0$, $\forall \alpha = 1, 2, \ldots, N$ and $a = 1, 2, 3$. However, the conclusion we reach can be generalized to more arbitrary initial conditions. Recall that $\mathcal{M}_s$ is the space of probability distributions $P(\vec{X}|\vec{\Theta})$ labeled by $6N$ statistical parameters $\vec{\Theta}$. These parameters are the coordinates for the point P, and in these coordinates a volume element $dV_{\mathcal{M}_s}$ reads,

$$dV_{\mathcal{M}_S} = \sqrt{g}d^{6N}\vec{\Theta} = \prod_{\alpha=1}^{N}\prod_{a=1}^{3}\frac{\sqrt{2}}{(\sigma_a^{(\alpha)})^2}d\mu_a^{(\alpha)}d\sigma_a^{(\alpha)} \tag{17}$$

where $g = |\det(g_{\mu\nu})|$. The volume of an extended region $\Delta V_{\mathcal{M}_s}(\tau; \lambda)$ of $\mathcal{M}_s$ is defined by,

$$\Delta V_{\mathcal{M}_s}(\tau; \lambda) \stackrel{\text{def}}{=} \prod_{\alpha=1}^{N}\prod_{a=1}^{3}\int_{\mu_a^{(\alpha)}(0)}^{\mu_a^{(\alpha)}(\tau)}\int_{\sigma_a^{(\alpha)}(0)}^{\sigma_a^{(\alpha)}(\tau)}\frac{\sqrt{2}}{(\sigma_a^{(\alpha)})^2}d\mu_a^{(\alpha)}d\sigma_a^{(\alpha)} \tag{18}$$

where $\mu_a^{(\alpha)}(\tau)$ and $\sigma_a^{(\alpha)}(\tau)$ are given in (16) and where the scalar λ is the chosen quantity to define the one-parameter family of geodesics $\mathcal{F}_{G_{\mathcal{M}_s}}(\lambda) \stackrel{\text{def}}{=} \{\Theta^{\mu}_{\mathcal{M}_s}(\tau; \lambda)\}_{\lambda \in \mathbb{R}^+}^{\mu=1,\ldots,6N}$. The quantity that encodes relevant information about the stability of neighboring volume elements is the average volume $\mathcal{V}_{\mathcal{M}_s}(\tau;\lambda)$ defined as [23, 24],

$$\mathcal{V}_{\mathcal{M}_s}(\tau;\lambda) \equiv \langle \Delta V_{\mathcal{M}_s}(\tau';\lambda)\rangle_\tau \stackrel{\text{def}}{=} \frac{1}{\tau}\int_0^\tau \Delta V_{\mathcal{M}_s}(\tau';\lambda)d\tau' \stackrel{\tau\to\infty}{\approx} e^{3N\lambda\tau}. \tag{19}$$

This asymptotic regime of diffusive evolution in (19) describes the exponential increase of average volume elements on $\mathcal{M}_s$. The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the statistical manifolds. From equation (19), we notice that the parameter λ characterizes the exponential growth rate of average statistical volumes $\mathcal{V}_{\mathcal{M}_s}(\tau;\lambda)$ in $\mathcal{M}_s$. This suggests that λ may play the same role ordinarily played by Lyapunov exponents. Indeed, it is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos where the *von Neumann entropy* increases linearly at a rate determined by the Lyapunov exponents. The linear entropy increase as a quantum chaos criterion was introduced by Zurek and Paz [17, 18]. In our information-geometric approach a relevant variable that can be useful to study the degree of instability characterizing the ED model is the IGE quantity defined as [23, 24],

$$\mathcal{S}_{\mathcal{M}_s} \stackrel{\text{def}}{=} \lim_{\tau\to\infty} \log \mathcal{V}_{\mathcal{M}_s}(\tau;\lambda). \tag{20}$$

The IGE is intended to capture the temporal complexity (chaoticity) of ED theoretical models on curved statistical manifolds $\mathcal{M}_s$ by considering the asymptotic temporal behaviors of the average statistical volumes occupied by the evolving macrovariables labelling points on $\mathcal{M}_s$.

Substituting (18) in (19), (20) becomes,

$$\mathcal{S}_{\mathcal{M}_s} = \lim_{\tau\to\infty} \log\left\{\frac{1}{\tau}\int_0^\tau \left[\prod_{\alpha=1}^{N}\prod_{a=1}^{3}\int_{\mu_a^{(\alpha)}(0)}^{\mu_a^{(\alpha)}(\tau')}\int_{\sigma_a^{(\alpha)}(0)}^{\sigma_a^{(\alpha)}(\tau')} \frac{\sqrt{2}}{(\sigma_a^{(\alpha)})^2} d\mu_a^{(\alpha)} d\sigma_a^{(\alpha)}\right] d\tau'\right\} \stackrel{\tau\to\infty}{\approx} 3N\lambda\tau. \tag{21}$$

Before discussing the meaning of (21), recall that in a rigorous examination of the entropy approach to the classical-quantum correspondence problem, Zurek and Paz consider the completely tractable model of an inverted harmonic oscillator with a potential $V(x) = -\frac{\Omega^2 x^2}{2}$ coupled to a high temperature (harmonic) bath. In their case, Ω is analogous to a Lyapunov exponent in a genuinely chaotic system. On calculating the rate of change of von Neumann entropy, they show that [17, 18]

$$\mathcal{S}_{\text{von Neumann}}(\tau) \stackrel{t\to\infty}{\approx} \Omega\tau \tag{22}$$

where $\mathcal{S}_{\text{von Neumann}}(\tau) = -tr(\rho_r(\tau)\log\rho_r(\tau))$ is the von Neumann entropy of the system [28] and $\rho_r(\tau)$ is the reduced density matrix of the system at time τ. The quantum entropy production rate is determined by the classical instability parameter Ω. Given that the classical Lyapunov exponent to which Ω is analogous is equal to the Kolmogorov-Sinai (KS) entropy of the system [28], this is indeed a remarkable characterization. It suggests that after a time, a quantum, classically chaotic system loses information to the environment at a rate determined entirely by the rate at which the classical system loses information as a result of its dynamics, namely, the KS entropy.

In analogy to the Zurek-Paz quantum chaos criterion in its classical reversible limit [29], we suggest a classical information-geometric criterion of linear IGE growth. The entropy-like quantity $\mathcal{S}_{\mathcal{M}_s}$ in (21) is the asymptotic limit of the natural logarithm of the statistical weight $\langle \Delta V_{\mathcal{M}_s}\rangle_\tau$ defined on $\mathcal{M}_s$ and it grows linearly in time, a quantum feature of chaos.

Indeed, equation (21) may be considered the information-geometric analog of the Zurek-Paz chaos criterion. In our chaotic ED Gaussian model, the IGE production rate is determined by the information-geometric parameter λ characterizing the exponential growth rate of average statistical volumes $\mathcal{V}_{\mathcal{M}_s}(\tau;\lambda)$ in $\mathcal{M}_s$.

3.3 Jacobi Field Intensity

Finally, we consider the behavior of the one-parameter family of neighboring geodesics $\mathcal{F}_{G_{\mathcal{M}_s}}(\lambda) \overset{\text{def}}{=} \{\Theta^{\mu}_{\mathcal{M}_s}(\tau;\lambda)\}^{\mu=1,\ldots,6N}_{\lambda\in\mathbb{R}^+}$ where,

$$\mu_a^{(\alpha)}(\tau;\lambda) = \frac{\frac{\Lambda^2}{2\lambda}}{\exp(-2\lambda\tau)+\frac{\Lambda^2}{8\lambda^2}}, \qquad \sigma_a^{(\alpha)}(\tau;\lambda) = \frac{\Lambda\exp(-\lambda\tau)}{\exp(-2\lambda\tau)+\frac{\Lambda^2}{8\lambda^2}} \tag{23}$$

with $\alpha = 1,2,\ldots,N$ and $a = 1,2,3$. The relative geodesic spread on a (non-maximally symmetric) curved manifold as $\mathcal{M}_s$ is characterized by the Jacobi-Levi-Civita equation, the natural tool to tackle dynamical chaos [21, 30],

$$\frac{D^2\delta\Theta^{\mu}}{D\tau^2} + R^{\mu}_{\nu\rho\sigma}\frac{\partial\Theta^{\nu}}{\partial\tau}\delta\Theta^{\rho}\frac{\partial\Theta^{\sigma}}{\partial\tau} = 0 \tag{24}$$

where the covariant derivative $\frac{D^2\delta\Theta^{\mu}}{D\tau^2}$ in (24) is defined as [31],

$$\begin{aligned}\frac{D^2\delta\Theta^{\mu}}{D\tau^2} &= \frac{d^2\delta\Theta^{\mu}}{d\tau^2} + 2\Gamma^{\mu}_{\alpha\beta}\frac{d\delta\Theta^{\alpha}}{d\tau}\frac{d\Theta^{\beta}}{d\tau} + \Gamma^{\mu}_{\alpha\beta}\delta\Theta^{\alpha}\frac{d^2\Theta^{\beta}}{d\tau^2} + \Gamma^{\mu}_{\alpha\beta,\nu}\frac{d\Theta^{\nu}}{d\tau}\frac{d\Theta^{\beta}}{d\tau}\delta\Theta^{\alpha}\\ &\quad + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\rho\sigma}\frac{d\Theta^{\sigma}}{d\tau}\frac{d\Theta^{\beta}}{d\tau}\delta\Theta^{\rho},\end{aligned} \tag{25}$$

and the Jacobi vector field J^{μ} is given by [26],

$$J^{\mu} \equiv \delta\Theta^{\mu} \overset{\text{def}}{=} \delta_{\lambda}\Theta^{\mu} = \left(\frac{\partial\Theta^{\mu}(\tau;\lambda)}{\partial\lambda}\right)_{\tau}\delta\lambda. \tag{26}$$

Equation (24) forms a system of $6N$ coupled ordinary differential equations *linear* in the components of the deviation vector field (26) but *nonlinear* in derivatives of the metric (7). It describes the linearized geodesic flow: the linearization ignores the relative velocity of the geodesics. When the geodesics are neighboring but their relative velocity is arbitrary, the corresponding geodesic deviation equation is the so-called generalized Jacobi equation [32]. The nonlinearity is due to the existence of velocity-dependent terms in the system. Neighboring geodesics accelerate relative to each other with a rate directly measured by the curvature tensor $R_{\alpha\beta\gamma\delta}$. Substituting (23) in (24) and neglecting the exponentially decaying terms in $\delta\Theta^{\mu}$ and its derivatives, integration of (24) leads to the following asymptotic expression of the Jacobi vector field intensity,

$$J_{\mathcal{M}_S} = \|J\| = (g_{\mu\nu}J^{\mu}J^{\nu})^{\frac{1}{2}} \overset{\tau\to\infty}{\approx} 3Ne^{\lambda\tau}. \tag{27}$$

Further details on the derivation of this result are in [23, 24]. We conclude that the geodesic spread on $\mathcal{M}_s$ is described by means of an *exponentially divergent* Jacobi vector field intensity $J_{\mathcal{M}_s}$, a *classical* feature of chaos. In our approach the quantity λ_J,

$$\lambda_J \overset{\text{def}}{=} \lim_{\tau\to\infty}\left[\frac{1}{\tau}\log\left(\left\|\frac{J_{\mathcal{M}_S}(\tau)}{J_{\mathcal{M}_S}(0)}\right\|\right)\right] \tag{28}$$

would play the role of the conventional Lyapunov exponents.

In conclusion, the main results of this work are encoded in the following equations,

$$\mathcal{R}_{\mathcal{M}_s} = -3N, \qquad \mathcal{S}_{\mathcal{M}_s} \overset{\tau\to\infty}{\approx} 3N\lambda\tau, \qquad J_{\mathcal{M}_S} \overset{\tau\to\infty}{\approx} 3Ne^{\lambda\tau}. \tag{29}$$

The IGE grows linearly as a function of the number of Gaussian-distributed microstates of the system. This supports the fact that $S_{\mathcal{M}_s}$ may be a useful measure of temporal complexity [33, 34]. Furthermore, these three indicators of chaoticity, the Ricci scalar curvature $\mathcal{R}_{\mathcal{M}_s}$, the information-geometric entropy $\mathcal{S}_{\mathcal{M}_s}$ and the Jacobi vector field intensity $J_{\mathcal{M}_S}$ are proportional to $3N$, the dimension of the microspace with microstates $\{\vec{X}\}$ underlying our chaotic ED Gaussian model. This proportionality leads to the conclusion that there is a substantial link among these information-geometric measures of chaoticity since they are all extensive functions of the dimensionality of the microspace underlying the macroscopic chaotic entropic dynamics, namely

$$\mathcal{R}_{\mathcal{M}_s} \sim \mathcal{S}_{\mathcal{M}_s} \sim J_{\mathcal{M}_S}. \tag{30}$$

Equation (30) represents the fundamental result of this work: curvature, information-geometric entropy and Jacobi field intensity are linked within our formalism. We are aware that our findings are reliable in the restrictive assumption of Gaussianity. However, we believe that with some additional technical machinery, more general conclusions can be achieved and this connection among indicators of chaoticity may be strengthened.

4 Conclusions

In this paper, a Gaussian ED statistical model has been constructed on a $6N$-dimensional statistical manifold $\mathcal{M}_s$. The macrocoordinates on the manifold are represented by the expectation values of microvariables associated with Gaussian distributions. The geometric structure of $\mathcal{M}_s$ was studied in detail. It was shown that $\mathcal{M}_s$ is a curved manifold of constant negative Ricci curvature $-3N$. The geodesics of the ED model are hyperbolic curves on $\mathcal{M}_s$. A study of the stability of geodesics on $\mathcal{M}_s$ was presented. The notion of statistical volume elements was introduced to investigate the asymptotic behavior of a one-parameter family of neighboring volumes $\mathcal{F}_{V_{\mathcal{M}_s}}(\lambda) \overset{\text{def}}{=} \{V_{\mathcal{M}_s}(\tau;\lambda)\}_{\lambda\in\mathbb{R}^+}$. An information-geometric analog of the Zurek-Paz chaos criterion was suggested. It was shown that the behavior of geodesics is characterized by exponential instability that leads to chaotic scenarios on the curved statistical manifold. These conclusions are supported by a study based on the geodesic deviation equations and on the asymptotic behavior of the Jacobi vector field intensity $J_{\mathcal{M}_s}$ on $\mathcal{M}_s$. A Lyapunov exponent analog similar to that appearing in the Riemannian geometric approach to chaos was suggested as an indicator of chaoticity. On the basis of our analysis a relationship among an entropy-like quantity, chaoticity and curvature is proposed, suggesting to interpret the statistical curvature as a measure of the entropic dynamical chaoticity. We think this is a relevant result since a rigorous relation among curvature, Lyapunov exponents and Kolmogorov-Sinai entropy is still under investigation and since there does not exist a well defined unifying characterization of chaos in classical and quantum physics due to fundamental differences between the two theories. Finally we remark that based on the results obtained from the chosen ED model, it is not unreasonable to think that should the correct variables describing the true degrees of freedom of a physical system be identified, perhaps deeper insights into the foundations of models of physics and reasoning (and their relationship to each other) may be uncovered.

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